Engineering Notes

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Comparison of Several Finite-Difference Methods

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Introduction

FINITE-DIFFERENCE methods for viscous and inviscid flows are prone to dissipation and dispersion errors. These are especially significant when the solution contains discontinuities either as shock waves or contact surfaces. In this Note, dissipation and dispersion errors in several high-order explicit finite-difference methods are presented based on wave-propagation and shock-tube problems and viscous Burgers' equation. The effects of smoothing operators are investigated, and their coefficients are optimized by numerical experimentation.

The first method employed in this Note is MacCormack's explicit method,[1] which is second order in space and time. Also implemented is the third-order, noncentered Warming-Kutler-Lomax (WKL) method,[2] which contains an adjustable parameter ω, whose value can be calculated in terms of the local eigenvalue (i.e., maximum total characteristic velocity) to obtain minimum dispersive error. The third scheme employed herein is the Two-Four (TF) scheme, which is second- and fourth-order accurate in time and space, respectively.[3,4] This method is inherently dissipative, and the degree of dissipation is controlled by an adjustable constant σ_m. When $\sigma_m = 7/9$, the method reduces to a form similar to the MacCormack scheme. In problems containing discontinuities, two types of numerical smoothing were used: MacCormack explicit fourth-order smoothing,[1] which has a free parameter α; the flux correction method of Boris and Book,[5] consisting of a diffusion step and an antidiffusion step adjusted by parameters η (arbitrary constant) and σ (damping coefficient). To the knowledge of the present authors, the applicability of this smoothing for finite-difference schemes has not been explored as yet.

Numerical Examples and Discussion

One-Dimensional Wave Propagation

The first example problem consists of the scalar equation for hyperbolic wave propagation, $W_t + W_x = 0$ with $W(x,0) = f(x)$. This problem is described by Gottlieb and Turkel with the initial conditions $f(x) = \sin(8\pi(x-1))$, $1 \leq x \leq 2$. The exact solution is a nondissipative wave moving to the right given as $W(x,t) = f(x-t)$, and the numerical domain of the integration is taken to be $0 \leq x \leq 20$. The numerical accuracy of the solution depends on the mesh size, Δx, and the time step, Δt, determined from the Courant-Friedrichs-Lewy (CFL) condition. A mesh spacing equal to $\Delta x = 0.01$ was found to be adequate from a mesh-refinement study. Several calculations with varying CFL were performed. For the MacCormack method, a reduction from CFL = 1.0 to 0.9 increases the dispersion error, whereas further reduction to CFL = 0.5 gives an inaccurate solution; here, CFL is equal to $\lambda \Delta t/\Delta x \leq 1$ and $\lambda = 1$. However, solutions obtained from the TF method suggest that CFL ≈ 0.2 gives the best result for this problem; if CFL is decreased to 0.1 or increased to 0.4, the dispersion errors increase significantly. The WKL method has significant amplitude errors for all of the CFL values and optimizes at CFL = 0.1.

Figure 1 shows the solution for each of these methods using the optimum Courant numbers listed in Table 1. Notice that the MacCormack scheme gives the best results. Small leading dispersive errors locally contaminate the TF method but amplitude distributions remain accurate. Finally, the WKL method presents very significant amplitude errors when dispersive errors are minimized. Consequently, it is feasible to assert that, due to its dissipative properties at high frequencies, the WKL method could introduce significant errors in problems involving wave interactions and wave propagation.

The flux-correction procedure was applied to the MacCormack and TF methods to determine the amplitude attenuation and high-frequency filtering induced by this smoothing operator. The optimized distribution for the TF method is shown in Fig. 2; parameter values are listed in Table 2. Since the minima and maxima of the distribution remain fairly sharp, it is likely that high-frequency oscillations will not be filtered out totally by this procedure. In addition, note that the application of this smoothing to the MacCormack method gave almost identical results.

Shock-Tube Problem

The second problem investigated in this work is the Sod example,[6] which involves the solution of the one-dimensional Euler system. Initial conditions at $t=0$ are defined as $\rho = 1.0$, $V = 0.0$, and $E = 2.5$ for $0 \leq x \leq ½$; $\rho = 0.125$, $V = 0.0$, and $E = 0.25$ for $½ \leq x \leq 1$. Here ρ is the density, V the velocity, and E the specific internal energy. The ratio of specific heats is $\gamma = 1.4$, and all of the results are given at $t = 0.25$, with $\Delta x = 0.01$. This system of equations is hyperbolic and the characteristic velocities are V, $V+c$, and $V-c$, where c is the local speed of sound in the gas. The numerical stability of the explicit methods used in this work are governed by the CFL condition, which ensures that $\Delta t/\Delta x(V+c) < 1$. This problem contains a rarefaction wave, a contact discontinuity, and a shock wave; therefore, this is an excellent test base to investigate numerical accuracy.

The numerical solution to this problem using the MacCormack and TF schemes requires a damping mechanism to capture the discontinuities. For this purpose, the fourth-order smoothing was applied to the MacCormack scheme. It is evident that this mechanism decreases dispersion errors using CFL = 1 and $\alpha = 0.1$ as the optimum values; however, significant errors still remain in the density distribution. When flux correction is applied to the MacCormack method, all of the dispersion errors disappear, and the dissipation errors in the rarefaction and shock waves are less than in the contact discontinuity, as shown in Figs. 3a and 3b. Here, the optimum parameter values listed in Table 2 are used. Next, flux correction is implemented with the TF method, which gives marginally better results than the MacCormack method, but still dis-

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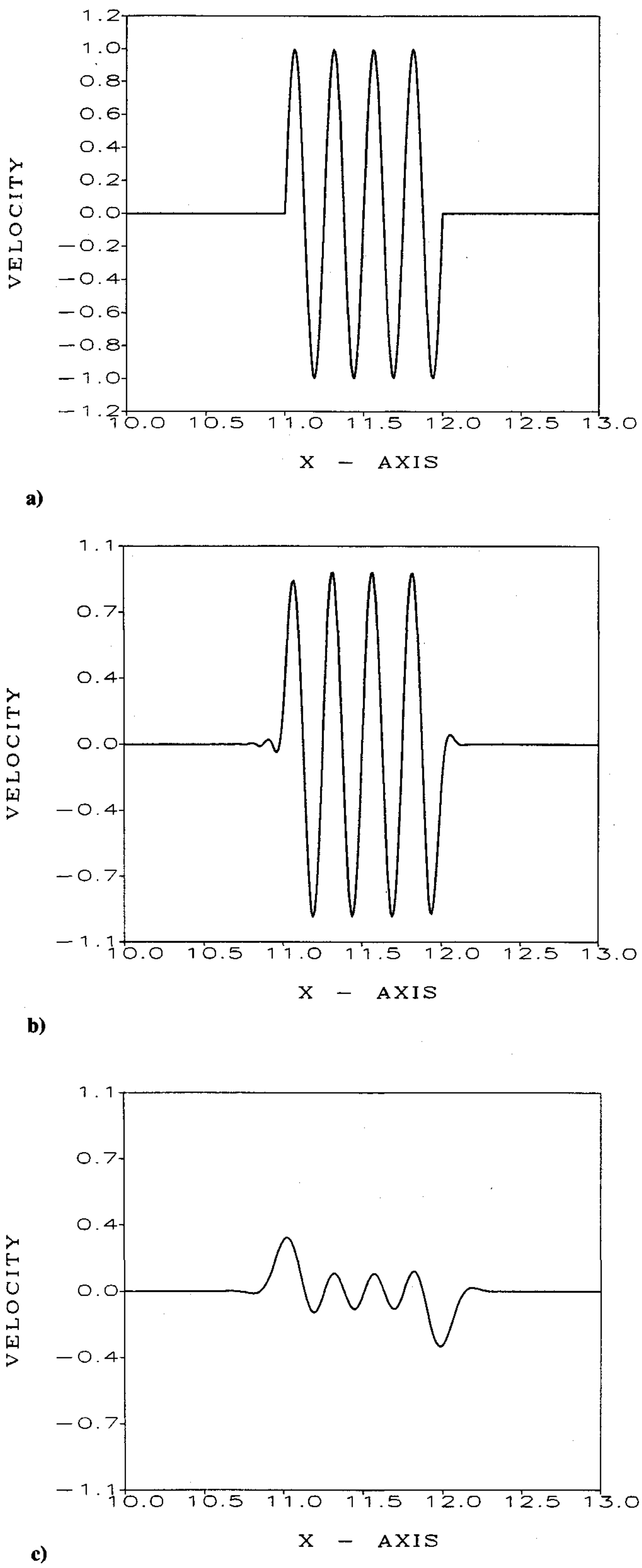

Fig. 1 Wave equation solution with $\Delta x = 0.01$: a) MacCormack method; b) TF method; c) WKL method.

plays dissipation errors appearing in rarefaction wave, contact discontinuity, and in the shock wave, as shown in Fig. 3c.

Figure 3d shows the WKL solution with the optimum CFL = 1.0. These results were obtained by using the value of ω calculated to minimize dispersion errors. Consequently, this error is very small at the head and tail of the rarefaction wave and at the contact discontinuity, as apparent in the density distribution. Furthermore, small dissipation and dispersion er-

Table 1 Optimum CFL values

Problem	MacCormack	WKL	TF
Wave	1.0	0.1	0.2
Shock tube	1.0	0.2	⅔
Viscous model	0.7	—	0.4

Table 2 Optimum values of CFL, α, σ, and η

	MacCormack				TF		
Problem	CFL	α	σ	η	CFL	σ	η
Wave	1.0	—	0.08	1/12	0.1	0.08	1/12
Shock tube	1.0	0.1	0.19	1/40	2/3	0.125	1/40
Viscous model	0.7	—	0.36	1/40	0.51	0.21	1/40

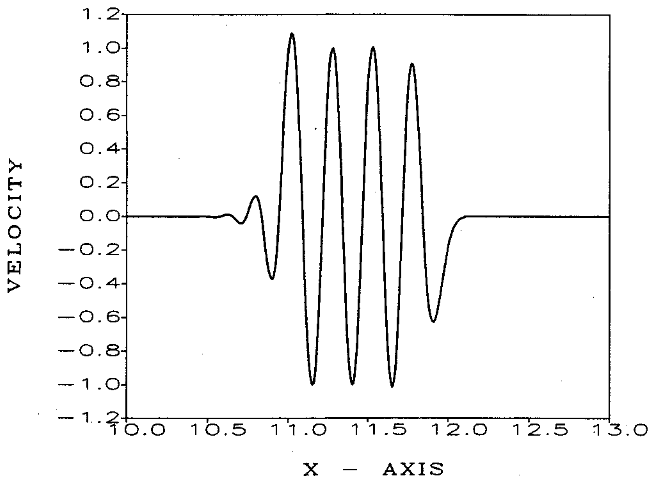

Fig. 2 Effect of flux correction on the TF method for the wave equation solution.

rors occur before and after the shock wave. In all of these methods, it was found that the errors increase as CFL decreases.

The preceding study, using various smoothing techniques, provides a means of isolating the effects of the damping coefficients α, σ, and η on problems containing discontinuities. For the various schemes tested in this work, the optimum values of CFL are presented in Table 1, and the optimum values of α, σ, and η in Table 2.

Viscous Burgers' Equation

In this section, the one-dimensional, nonlinear, viscous Burgers' equation with $-1 \leq x \leq 1$; $\upsilon = 0.01/\pi$ is considered. The initial condition is given as $U(x, 0) = -\sin(\pi x)$, and the boundary conditions are $U(1, t) = U(-1, t) = 0$. This problem develops a very steep gradient in the center of the domain, with a maximum slope at $x = 0.0$, and $t = 0.5$. For this problem, $\Delta x = 0.01$ was used and only the MacCormack and TF methods were considered. As expected, in the absence of numerical smoothing, the time evolution of the solutions obtained using both methods displayed strong dispersive errors leading the shock; these errors were reduced significantly by optimizing the Courant number, but the need for artificial smoothing remained evident. Therefore, flux correction was applied to each of these methods. In Fig. 4, the results for the TF method are presented with time increments of 0.1, starting with the initial conditions and up to $t = 0.5$ (the solutions for the MacCormack method were virtually identical). From these results, it is apparent that the dispersive errors are well controlled and the amplitude attenuation remains small.

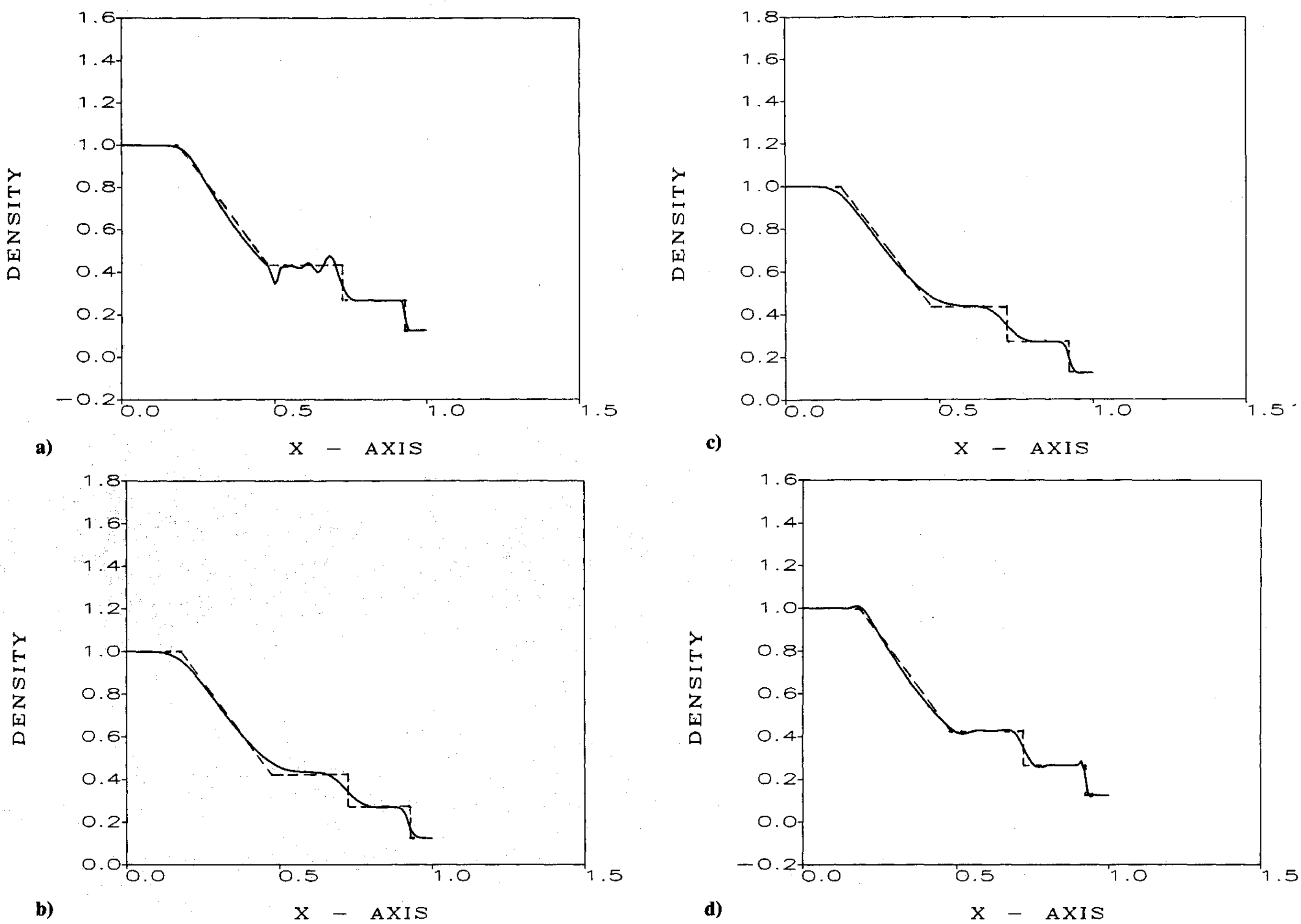

Fig. 3 Shock-tube problem, density profiles: a) MacCormack method with fourth-order smoothing; b) MacCormack method with flux correction; c) TF method with flux corrections; d) WKL method. Parameter values are listed in Tables 1 and 2.

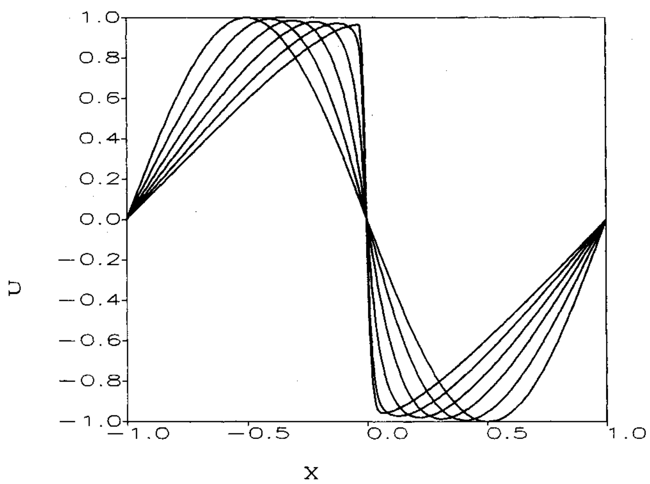

Fig. 4 Viscous model equation solution with flux correction using the TF method. Parameter values are listed in Table 2.

Concluding Remarks

This numerical study indicates that with the MacCormack and TF methods, solution accuracy for different problems involving wave propagation, shock-wave and contact discontinuities, and viscous effects is strongly dependent on the Courant number. The application of flux correction to the MacCormack and TF methods significantly attenuates dispersion errors, and the ensuing solutions capture the discontinuities in the shock-tube problem with improved accuracy and resolution. Finally, the flux-corrected MacCormack and Two-Four methods are free of dispersion errors for the viscous Burgers' equation at the expense of small dissipation errors.

Acknowledgment

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